

# P-symbols, Heun identities, and ${}_3F_2$ identities

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**ABSTRACT.** The usefulness of Riemann P-symbols in deriving identities involving the parametrized special function  $Hl$  is explored.  $Hl$  is the analytic local solution of the Heun equation, the canonical second-order differential equation on the Riemann sphere with four regular singular points. The identities discussed include ones coming from Möbius automorphisms and F-homotopies, and also quadratic and biquadratic transformations. The case when  $Hl$  is identical to a generalized hypergeometric function of  ${}_3F_2$  type is examined, and Pfaff and Euler transformations of  ${}_3F_2(a_1, a_2, e + 1; b_1, e; x)$  are derived. They extend several  ${}_3F_2$  identities of Bailey and Slater.

## 1. Introduction

The Gauss hypergeometric equation (GHE) and the Heun equation (HE) [18] are canonical second-order Fuchsian differential equations on the Riemann sphere  $\mathbb{P}_x^1$  with three and four singular points, respectively. (A differential equation is Fuchsian if each of its singular points is regular.) By convention, they are written as

$$(1a) \quad \left\{ D^2 + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} \right] D + \left[ \frac{\alpha\beta}{x(x-1)} \right] \right\} F = 0,$$

$$(1b) \quad \left\{ D^2 + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right] D + \left[ \frac{\alpha\beta x - q}{x(x-1)(x-a)} \right] \right\} F = 0,$$

respectively. Here  $\alpha, \beta; \gamma, \delta$ , resp.  $a (\neq 0, 1), q; \alpha, \beta; \gamma, \delta, \epsilon$  are complex-valued parameters; and each equation is invariant under  $\alpha \leftrightarrow \beta$ . The singular points are  $x = 0, 1, \infty$ , resp.  $x = 0, 1, a, \infty$ , and their characteristic exponents are  $0, 1 - \gamma; 0, 1 - \delta; \alpha, \beta$ , resp.  $0, 1 - \gamma; 0, 1 - \delta; 0, 1 - \epsilon; \alpha, \beta$ . These parameters are constrained by Fuchs's relation on exponents. The sum of the exponents of the GHE, resp. the HE, must be 1, resp. 2. So  $\delta = \alpha + \beta - \gamma + 1$ , resp.  $\epsilon = \alpha + \beta - \gamma - \delta + 1$ . If this Fuchsian condition does not hold, the singular point  $x = \infty$  will be irregular. The parameter  $q \in \mathbb{C}$  of the HE is 'accessory': it does not affect the exponents.

If  $\gamma$  is a nonpositive integer, the local (Frobenius) solution of the GHE or HE corresponding to the zero exponent at  $x = 0$  will generically be logarithmic; we do not consider this case. If  $\gamma$  is *not* a nonpositive integer, the solution will be analytic at  $x = 0$ . When normalized to unity at  $x = 0$ , it is denoted  ${}_2F_1(x) := {}_2F_1(\alpha, \beta; \gamma; x)$ ,

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resp.  $Hl(x) := Hl(a, q; \alpha, \beta; \gamma, \delta; x)$ , and is called the Gauss hypergeometric function, resp. the local Heun function. (The shorter term ‘Heun function’ has a more specialized meaning [18].) The redundant parameter  $\delta$ , resp.  $\epsilon$ , is suppressed.

The parametrized special functions  ${}_2F_1, Hl$  have series expansions of the form  $\sum_{n=0}^{\infty} c(n)x^n$ . The coefficient sequences  $c: \mathbb{N} \rightarrow \mathbb{C}$  satisfy respective recurrences

$$(2a) \quad \left\{ (n + \gamma)(n + 1)E - (n + \alpha)(n + \beta) \right\} c = 0,$$

$$(2b) \quad \left\{ (n + \gamma + 1)(n + 2)aE^2 - [(n + 1)(n + \gamma + \delta)a + (n + 1)(n + \gamma + \epsilon) + q]E + (n + \alpha)(n + \beta) \right\} c = 0,$$

where  $E$  denotes the shift operator, i.e.,  $(E^k c)(n) := c(n + k)$ . The recurrence (2a), resp. (2b), is initialized by  $c(0) = 1$ , resp.  $c(0) = 1, c(-1) = 0$ . If  $c(n)$  is taken to equal zero when  $n < 0$ , then (2a),(2b) will hold for all  $n \in \mathbb{Z}$ . The power series converges on  $|x| < 1$ , resp.  $|x| < \min(1, |a|)$ .

The special functions  ${}_2F_1, Hl$  play a central role not only in the theory of second-order Fuchsian differential equations, but also in the theory of linear recurrences with quadratic coefficients. From (2a),(2b), one easily deduces the following.

PROPOSITION 1.1.

- (1) Suppose  $c: \mathbb{N} \rightarrow \mathbb{C}$  (with  $c(n) := 0$  for  $n < 0$  by convention) satisfies a 2-term recurrence  $[P_1(n)E + P_0(n)]c = 0$ , where  $\deg P_1 = \deg P_0 = 2$ , and one root of  $P_1$  is  $-1$  and the other is not an integer greater than  $-1$ . Then the ordinary generating function (o.g.f.) of  $c$  equals  $C {}_2F_1(\alpha, \beta; \gamma; Ax)$  for some  $A(\neq 0), C \in \mathbb{C}$  and some choice of GHE parameters  $\alpha, \beta, \gamma$ .
- (2) Suppose  $c: \mathbb{N} \rightarrow \mathbb{C}$  (with  $c(n) := 0$  for  $n < 0$  by convention) satisfies a 3-term recurrence  $[P_2(n)E^2 + P_1(n)E + P_0(n)]c = 0$ , where  $\deg P_2 = \deg P_0 = 2$ ,  $\deg P_1 \leq 2$ , and one root of  $P_2$  is  $-2$  and the other is not an integer greater than  $-2$ . Suppose, moreover, that the characteristic polynomial of this recurrence has distinct roots. (This polynomial is formed from the coefficients of  $n^2$  in  $P_2, P_1, P_0$ : it equals  $p_{2,2}n^2 + p_{1,2}n + p_{0,2}$ .) Then the o.g.f. of  $c$  equals  $C Hl(a, q; \alpha, \beta; \gamma, \delta; Ax)$  for some  $A(\neq 0), C \in \mathbb{C}$  and some choice of HE parameters  $a, q; \alpha, \beta; \gamma, \delta$ .

In part (2), the taking of the polynomial roots to be distinct avoids the case when the differential equation satisfied by the o.g.f. is confluent, i.e., non-Fuchsian.

We explain here how the classical formalism of Riemann P-symbols (see §2) facilitates the derivation of *Heun identities*, such as alternative expressions for the local function  $Hl$ , written in terms of itself. In §3, we begin by determining the automorphism groups of the GHE and HE. We showed in a previous treatment [14] that the two automorphism groups, of orders 24 and 192 respectively, are the Coxeter groups  $\mathcal{D}_3 (\cong S_4)$  and  $\mathcal{D}_4$ . (The set of 24 GHE automorphisms was first worked out by Kummer, but its group structure long remained obscure; an isomorphism to the octahedral group  $S_4$  was pointed out by Dwork [7] and others [5, 12].)  $\mathcal{D}_3$  and  $\mathcal{D}_4$  have subgroups  $\mathcal{D}_2$  and  $\mathcal{D}_3$ , which are the transformation groups of  ${}_2F_1$  and  $Hl$ . The former, of order 4, is generated by classical transformations of Pfaff and Euler, but the latter, of order 24, is a novel object. In [14], we computed the 24 resulting expressions for  $Hl(x)$  with the aid of a computer algebra system. We show here how with the aid of P-symbols, they may be computed by hand.

In § 4, we compute quadratic and quartic transformations of  $Hl$ , which express  $Hl(x)$  in terms of  $Hl(R(x))$ , where  $R$  is rational of degree 2 or 4. It is known that the derivation of the quadratic transformations of  ${}_2F_1$  is facilitated by the P-symbol formalism [2, §3.9]. Erdélyi [8, §15.3] pointed out that it can also be used to derive quadratic and quartic HE transformations, but did not compute explicit formulae involving  $Hl$ . Up to automorphism, we find one quadratic transformation of  $Hl$  and one quartic one that is biquadratic, i.e., is the composition of two quadratics. The  $Hl$  situation apparently contrasts with that of  ${}_2F_1$ : not all the quartic transformations of  ${}_2F_1$ , which were worked out by Goursat [9], are biquadratic.

Using P-symbols, we investigate in § 5 the interesting case when  $Hl$  coincides with a generalized hypergeometric function, in particular with a  ${}_3F_2$  of the form  ${}_3F_2(a_1, a_2, e + 1; b_1, e; x)$ . If the parameters  $a, q; \alpha, \beta; \gamma, \delta$  are suitably chosen, this phenomenon will occur, as noted by Letessier, Valent, and Wimp [11]. Since the coefficients of the series expansion of  ${}_3F_2$  satisfy a first-order recurrence, the equality is attributable to a factorization of the second-order difference operator in (2b). Such factorizations can be computed algorithmically [6, 16]. We give an explicit one, and in the same spirit, factor the third-order differential operator in the equation satisfied by the  ${}_3F_2$ . The operator products in the two factorizations can be viewed as *desingularizations* in the sense of Abramov, Barkatou, and van Hoeij [1].

In § 6, we derive transformations analogous to Pfaff's and Euler's for the function  ${}_3F_2(a_1, a_2, e + 1; b_1, e; x)$ , involving nonlinear parametric constraints. Bailey and Slater (see, e.g., Slater [19, § 2.4.2]) found an Pfaff-like identity satisfied by  ${}_3F_2$ , with three free parameters, but our identity, which extends theirs, has four. As a corollary, we extend other results of Bailey; e.g., his result that any very well poised  ${}_3F_2$  can be written in terms of a well poised  ${}_2F_1$ , and vice versa.

## 2. Riemann P-symbols

Any P-symbol tabulates the characteristic exponents of a linear (homogeneous) Fuchsian differential equation (FDE)  $\mathcal{L}u = 0$  on  $\mathbb{P}^1$ . It partially characterizes the space of local solutions. The columns of the P-symbol list the exponents associated to each singular point of  $\mathcal{L}$ . For instance, the GHE and HE have P-symbols

$$(3) \quad \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & x \\ 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & \beta & \end{array} \right\}, \quad \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{array} \right\},$$

where the column and exponent orderings are not significant. Ordinary (non-singular) points may optionally be included, each with its own column. Any finite ordinary point of a second-order FDE has exponents 0, 1, and the point  $x = \infty$ , if ordinary, has exponents 0,  $-1$ . Having 'ordinary' exponents is not sufficient for ordinarity, since zero-exponent Frobenius solutions may be logarithmic.

Any second-order FDE on  $\mathbb{P}^1$  with  $n \geq 3$  singular points is determined up to equivalence by its  $2n$  exponents and  $n - 3$  accessory parameters. Though the latter parameters are not shown, P-symbols facilitate the computation of transformed FDEs, as the following indicates. Automorphisms of  $\mathbb{P}^1$  are Möbius transformations (also called homographies or linear fractional transformations) of the form  $M(x) = (Ax + B)/(Cx + D)$ , where  $A, B, C, D \in \mathbb{C}$  and  $AD \neq BC$ . Under such transformations, singular points are accompanied by their exponents. That is, if  $M: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$ , applying the change of variable  $x' = M(x)$  to an FDE  $\mathcal{L}u = 0$  on  $\mathbb{P}_x^1$ ,

lifts it to  $(M^*\mathcal{L})v = 0$  on  $\mathbb{P}_x^1$ , where if  $\mathcal{L}$  has singular points  $a', b', c', \dots$ , with respective exponents  $\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1, \gamma_2, \dots$ , the operator  $M^*\mathcal{L}$  will have P-symbol

$$(4) \quad \left\{ \begin{array}{cccc|c} M^{-1}a' & M^{-1}b' & M^{-1}c' & \dots & x \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & \end{array} \right\} = \left\{ \begin{array}{cccc|c} a' & b' & c' & \dots & M(x) \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & \end{array} \right\}.$$

Fuchs's relation states that the sum of the  $2n$  exponents of any second-order FDE with  $n$  singular points equals  $n - 2$ . Möbius lifting trivially preserves this.

Since there is a (unique) Möbius transformation that takes any three distinct points to any other three, one can normalize any FDE with  $n \geq 3$  singular points by moving three of them to  $0, 1, \infty$ . Multiplying the dependent variable by  $(x - x_0)^{-\zeta}$  will decrement the exponents at  $x = x_0$  by  $\zeta$ , and those at  $x = \infty$  by  $-\zeta$ ; i.e.,

$$(5) \quad (x - x_0)^{-\zeta} \left\{ \begin{array}{ccc|c} x_0 & \dots & \infty & x \\ \theta_1 & \dots & \eta_1 & \\ \theta_2 & \dots & \eta_2 & \end{array} \right\} = \left\{ \begin{array}{ccc|c} x_0 & \dots & \infty & x \\ \theta_1 - \zeta & \dots & \eta_1 + \zeta & \\ \theta_2 - \zeta & \dots & \eta_2 + \zeta & \end{array} \right\}$$

Using (5), one can further normalize the FDE by shifting one exponent at each finite singular point to zero. For  $n = 3, 4$ , the resulting FDE is the GHE, resp. HE.

Any nonconstant rational function  $x' = R(x)$  specifies a  $(\deg R)$ -sheeted covering  $R: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$ . Applying the change of variables  $x' = R(x)$  to an FDE  $\mathcal{L}u = 0$  on  $\mathbb{P}_{x'}^1$ , lifts it to an FDE  $(R^*\mathcal{L})v = 0$  on  $\mathbb{P}_x^1$ .

**PROPOSITION 2.1.** (i) *The characteristic exponents of  $R^*\mathcal{L}$  at any point  $x_0 \in R^{-1}x'_0$  equal those of  $\mathcal{L}$  at  $x'_0$ , multiplied by the multiplicity with which  $x_0$  is mapped to  $x'_0$ .* (ii) *If the exponents of  $\mathcal{L}$  at  $x'_0$  are  $0, 1/k$  for some integer  $k > 1$ , and  $x_0$  is mapped to  $x'_0$  with multiplicity  $k$ , then  $x_0$  will be an ordinary point of  $R^*\mathcal{L}$ .*

**PROOF.** Part (i) follows from the definition of a characteristic exponent. To prove (ii), it suffices to show that the zero-exponent Frobenius solution of  $(R^*\mathcal{L})v = 0$  at  $x = x_0$  cannot be logarithmic. By the definition of a lifting, this solution  $v = v(x)$  (normalization being arbitrary) can be written locally as  $u(R(x))$ , where  $u = u(x)$  is a zero-exponent Frobenius solution of  $\mathcal{L}u = 0$ . But such a solution can be logarithmic only if the other exponent is a positive integer; and here it is  $1/k$ .  $\square$

By examination, lifting  $\mathcal{L}$  to  $R^*\mathcal{L}$  preserves Fuchs's relation. But under a non-Möbius rational lifting, singular points may disappear or appear. Part (ii) of the proposition indicates how an inverse image of a singular point may fail to be singular. Also, if  $x'_0$  is an ordinary point of  $\mathcal{L}$ , with  $x_0 \in R^{-1}x'_0$  mapped by  $R$  to  $x'_0$  with multiplicity  $k > 1$ , then  $x_0$  will be a singular point of  $R^*\mathcal{L}$ , with exponents  $0, k$ .

With the aid of Prop. 2.1, one can conjecture and verify rational transformation laws for the special functions  ${}_2F_1$  and  $Hl$ . However, in any such identity the accessory parameter  $q$  of the HE will need to be considered separately. It should be mentioned that there are types of Heun identity, such as differential transformations, on which P-symbols throw less light. One example is the following.

**THEOREM 2.2** ([20], §3). *For every integer  $N \geq 0$ , one has the differential Heun identity*

$$D^N Hl(a, q; 1 - N, \beta; \gamma, \delta; x) \propto Hl(a, q'; 1 + N, \beta + N; \gamma + N, \delta + N; x),$$

where  $q' := q + N(N - 1)(a + 1) + N[(a + 1)\gamma + a\delta + \epsilon]$ .

This result can be strengthened to an equality between HE P-symbols, namely

$$(6) \quad D^N \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & 1-N & \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{array} \right\} = \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & 1+N & \\ 1-(\gamma+N) & 1-(\delta+N) & 1-(\epsilon+N) & \beta+N & \end{array} \right\},$$

which says that any local solution of the left-hand HE, differentiated  $N$  times, is one of the right-hand HE. The P-symbol identity (6) is plausible, since the exponents on left and right formally correspond. For instance, at  $x = 0$  the left-hand Frobenius solutions are of the form  $x^0$  and  $x^{1-\gamma}$  times analytic functions, each nonzero at  $x = 0$ ; so (generically only!), their  $N$ 'th derivatives are of the form  $x^0$  and  $x^{1-(\gamma+N)}$  times such analytic functions. But a satisfactory proof of (6) along these lines would require a study of the series coefficients of the Frobenius solutions, since if any non-leading coefficient were zero, it would yield non-generic behavior. And of course,  $q'$  needs to be computed separately. In the following sections the focus is on non-differential identities, which are easier to conjecture and prove.

### 3. Automorphisms of the GHE and HE

P-symbols of normalized FDEs with  $n \geq 3$  singular points can be written in a symmetrical way. Without loss of generality, the singular points may be taken to include  $x = 0, \infty$ ; and typically,  $x = 1$  as well. Each finite singular point may be taken to have a zero exponent. The P-symbol of such an FDE will be of the form

$$(7) \quad \left\{ \begin{array}{cccc|c} a_1 & a_2 & \dots & a_{n-1} & \infty & x \\ 0 & 0 & \dots & 0 & \alpha & \\ \theta_1 & \theta_2 & \dots & \theta_{n-1} & \beta := \alpha + \theta_\infty & \end{array} \right\},$$

where  $a_1 = 0$  (and usually,  $a_2 = 1$ ). The canonical FDE solution  $F$  at  $x = a_1 = 0$  will be written  $F(a_2, \dots, a_{n-1}; q; \theta_1, \dots, \theta_{n-1}, \theta_\infty; x)$ , where  $q$  is a suitably defined vector of  $n-3$  accessory parameters. Necessarily  $\alpha = (n-2-\theta_1-\dots-\theta_{n-1}-\theta_\infty)/2$ .

The automorphism group of such normalized FDEs, or P-symbols, contains a subgroup of automorphisms that negate exponents, or more accurately exponent *differences*. These are called F-homotopies [18]. The subgroup is generated by

$$(x-a_i)^{-\theta_i} \left\{ \begin{array}{cccc|c} \dots & a_i & \dots & \infty & x \\ \dots & 0 & \dots & \alpha & \\ \dots & \theta_i & \dots & \alpha + \theta_\infty & \end{array} \right\} = \left\{ \begin{array}{cccc|c} \dots & a_i & \dots & \infty & x \\ \dots & 0 & \dots & \alpha + \theta_i & \\ \dots & -\theta_i & \dots & \alpha + \theta_\infty + \theta_i & \end{array} \right\},$$

$i = 1, \dots, n-1$ , and by the trivial  $\alpha \leftrightarrow \beta$  interchange at  $x = \infty$ , i.e.,

$$\left\{ \begin{array}{cccc|c} a_1 & \dots & a_{n-1} & \infty & x \\ 0 & \dots & 0 & \alpha & \\ \theta_1 & \dots & \theta_{n-1} & \alpha + \theta_\infty & \end{array} \right\} = \left\{ \begin{array}{cccc|c} a_1 & \dots & a_{n-1} & \infty & x \\ 0 & \dots & 0 & \alpha + \theta_\infty & \\ \theta_1 & \dots & \theta_{n-1} & \alpha & \end{array} \right\}.$$

The group of F-homotopies is isomorphic to  $(\mathbb{Z}_2)^n$ ; or  $(\mathbb{Z}_2)^{n-1}$ , if  $\alpha \leftrightarrow \beta$  is excluded.

An F-homotopy at any of the  $n-1$  singular points other than  $x = a_1 = 0$  yields a transformation of the canonical local solution  $F = F(x)$  at  $x = a_1 = 0$ . Taking

the usual normalization  $F(x = a_1 = 0) = 1$  into account, one deduces

$$(8) \quad F(a_2, \dots, a_{n-1}; q; \theta_1, \dots, \theta_i, \dots, \theta_{n-1}, \theta_\infty; x) \\ = (1 - x/a_i)^{\theta_i} F(a_2, \dots, a_{n-1}; q'; \theta_1, \dots, -\theta_i, \dots, \theta_{n-1}, \theta_\infty; x),$$

$i = 2, \dots, n-1$ , where for each  $i$ , the transformed vector  $q'$  of accessory parameters must be computed separately, by manipulating the FDE. One also has

$$(9) \quad F(a_2, \dots, a_{n-1}; q; \theta_1, \dots, \theta_{n-1}, \theta_\infty; x) \\ = F(a_2, \dots, a_{n-1}; q; \theta_1, \dots, \theta_{n-1}, -\theta_\infty; x).$$

The group of F-homotopic transformations of  $F$  is isomorphic to  $(\mathbb{Z}_2)^{n-1}$ ; or to  $(\mathbb{Z}_2)^{n-2}$ , if the trivial transformation (9), i.e.,  $\alpha \leftrightarrow \beta$ , is excluded.

The GHE case  $n = 3$ , in which there are no accessory parameters, illustrates this. The identity (8), when  $n = 3$  and  $i = 2$ , says that

$$(10) \quad {}_2F_1(\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; x),$$

and (9) simply says that  ${}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(\beta, \alpha; \gamma; x)$ . Equation (10) is Euler's transformation of  ${}_2F_1$  [2], which is an involution that commutes with  $\alpha \leftrightarrow \beta$ .

The automorphism group of normalized FDEs  $\mathcal{L}u = 0$  with  $n$  singular points, or strictly of their P-symbols, also contains a subgroup of Möbius automorphisms, based on Möbius transformations. If  $\mathcal{L}u = 0$  is to be lifted to  $(M^*\mathcal{L})v = 0$  along  $M: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$ , the inverse images of the singular points  $x' = 0, 1, \infty$  of  $L$  must be chosen to be singular points of  $M^*\mathcal{L}$ . So there are  $n(n-1)(n-2)$  possibilities, and there is no additional freedom in the choice of  $M$ . But since the order of the remaining  $n-3$  columns in the lifted P-symbol is not significant, one may permute them arbitrarily. So the Möbius subgroup is isomorphic to the symmetric group  $S_n$ .

The cases  $n = 3, 4$  illustrate this. In the  $n = 3$  GHE case,  $x' = M(x)$  may be any of  $3 \cdot 2 \cdot 1 = 6$  homographies. The correspondence to permutations of  $0, 1, \infty$  is

$$\begin{array}{cccccc} x, & \frac{x}{x-1}, & 1-x, & \frac{x-1}{x}, & \frac{1}{x}, & \frac{1}{1-x} \\ (0)(1)(\infty), & (0)(1\infty), & (01)(\infty), & (0\infty 1), & (0\infty)(1), & (01\infty). \end{array}$$

In the  $n = 4$  HE case,  $x' = M(x)$  may be any of  $4 \cdot 3 \cdot 2 = 24$  homographies, given in [14]. The six that stabilize  $x = 0$ , which are of primary interest here, are

$$\begin{array}{cccccc} x, & \frac{x}{x-1}, & \frac{(1-a)x}{x-a}, & \frac{(a-1)x}{a(x-1)}, & \frac{x}{a}, & \frac{x}{x-a} \\ (0)(1)(a)(\infty), & (0)(1\infty)(a), & (0)(1)(a\infty), & (0)(1\infty a), & (0)(1a)(\infty), & (0)(1a\infty) \\ a, & \frac{a}{a-1}, & 1-a, & \frac{a-1}{a}, & \frac{1}{a}, & \frac{1}{1-a}. \end{array}$$

Each  $M$  corresponds to a permutation of the HE singular points  $0, 1, a, \infty$ , i.e., columns, as shown. But the permutation is to be interpreted in a special way. For instance,  $(0)(1a\infty)$  means  $0 \mapsto 0$ ,  $1 \mapsto a'$ ,  $a \mapsto \infty$ ,  $\infty \mapsto 1$ , for some  $a'$ . In general,  $a \in \mathbb{P}_x^1$  and  $a' \in \mathbb{P}_{x'}^1$  differ. The value of  $a'$ , in terms of  $a$ , is given in the third row.

A Möbius automorphism based on a lifting along  $x' = M(x)$  must in general include an exponent shift, of the type shown in (5), in order to keep one exponent zero at each finite singular point. The formulas for the Möbius transformations of the local solution  $F = F(x)$  at  $x = 0$  will include such factors. Any of the above  $n(n-1)(n-2)$  Möbius transformations that fixes the point  $x = a_1 = 0$  will yield a transformation of  $F$ . If the map  $x' = M(x)$  is not of the form  $x' = Ax$ , it must

be of the form  $M(x) = Ax/[x - M^{-1}(\infty)]$ , where  $M^{-1}(\infty)$  is one of  $a_2, \dots, a_{n-1}$ . Taking  $F(x = a_1 = 0) = 1$  into account, one obtains a transformation of the type

$$(11) \quad F(a_2, \dots, a_{n-1}; q; \theta_1, \dots, \theta_{n-1}, \theta_\infty; x) \\ = [1 - x/M^{-1}(\infty)]^{-\alpha} F(a'_2, a'_3, \dots, a'_{n-1}; q'; \theta'_1, \dots, \theta'_{n-1}, \theta'_\infty; M(x)),$$

in which the transformed vector  $q'$  of accessory parameters must be computed separately. Any such formula is really  $(n-3)!$  formally distinct transformations of  $F$ , since the  $n-3$  parameter pairs  $(a_3, \theta_3), \dots, (a_{n-1}, \theta_{n-1})$  on the left-hand side may be permuted arbitrarily. (Equivalently, the columns of the left-hand P-symbol, other than the  $x = 0, 1, \infty$  columns, may be permuted.) The group of Möbius transformations of  $F$  is isomorphic to  $S_{n-1}$ .

The  $n = 3$  GHE case is illustrative. Of the  $3 \cdot 2 \cdot 1 = 6$  homographies above, the only two that fix  $x = 0$  are the identity  $x \mapsto x$  and  $x \mapsto \frac{x}{x-1}$ ; they make up a group isomorphic to  $S_2$ . By examination, the latter yields the identity

$$(12) \quad {}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} {}_2F_1(\alpha, \gamma - \beta; \gamma; \frac{x}{x-1}),$$

valid near  $x = 0$ . This is Pfaff's transformation of  ${}_2F_1$  [2], which is an involution.

Any Möbius automorphism permutes the columns of the P-symbol, and any F-homotopic one acts on columns individually. So the  $S_n$  subgroup of the former normalizes the  $(\mathbb{Z}_2)^n$  subgroup of the latter, and the full group of automorphisms is isomorphic to a semidirect product  $(\mathbb{Z}_2)^n \rtimes S_n$ . It is easy to see that this is the wreath product  $\mathbb{Z}_2 \wr S_n$ , the group of signed permutations of an  $n$ -set. Any element of  $\mathbb{Z}_2 \wr S_n \cong (\mathbb{Z}_2)^n \rtimes S_n$  is an ordered pair  $(h, \sigma)$ , comprising a Möbius and an F-homotopic automorphism, which act from right to left. The first permutes the singular points, i.e., columns of the P-symbol, according to  $\sigma \in S_n$ , and the second negates zero or more of the exponent differences  $\theta_1, \dots, \theta_n$ . Any element  $(h, \sigma)$  can accordingly be written in a sign-annotated version of disjoint cycle notation. For instance,  $[0_+][1_+ a_- \infty_-]$  signifies the signed permutation  $0 \mapsto 0$ ,  $1 \mapsto a'$ ,  $a \mapsto \infty$ ,  $\infty \mapsto 1$ , followed by negations at  $x = \infty$  and  $x = 1$ . 'Even-signed' permutations, such as this one, include an even number of minus signs; they play a special role.

**THEOREM 3.1.** *The automorphism group of normalized second-order FDEs on  $\mathbb{P}^1$  with  $n \geq 3$  singular points is the Coxeter group  $\mathcal{B}_n := \mathbb{Z}_2 \wr S_n$ , the group of signed permutations of an  $n$ -set. If the transformation  $\alpha \leftrightarrow \beta$  is not explicitly included as an element, the automorphism group will be the Coxeter group  $\mathcal{D}_n := [\mathbb{Z}_2 \wr S_n]_{\text{even}}$ , the group of even-signed permutations of an  $n$ -set. The transformation group of the canonical solution  $F = F(x)$  is  $\mathcal{B}_{n-1}$ , or  $\mathcal{D}_{n-1}$  if  $\alpha \leftrightarrow \beta$  is not explicitly included.*

**REMARK.** The orders of these four groups are  $|\mathcal{B}_n| = 2^n n!$ ,  $|\mathcal{D}_n| = 2^{n-1} n!$ ,  $|\mathcal{B}_{n-1}| = 2^{n-1} (n-1)!$ , and  $|\mathcal{D}_{n-1}| = 2^{n-2} (n-2)!$ . It is readily verified that  $\mathcal{B}_3, \mathcal{D}_3$  are isomorphic to  $\mathbb{Z}_2 \times S_4$  and  $S_4$ ; and that  $\mathcal{B}_2, \mathcal{D}_2$  are isomorphic to the dihedral group of order 8 and the Klein four-group  $(\mathbb{Z}_2)^2$ . The groups  $\mathcal{B}_n, \mathcal{D}_n$  are usually not defined as permutation groups, but as reflection groups consistent with certain Coxeter graphs [10]. Due to this, the notation used here is slightly nonstandard: our  $\mathcal{D}_3$  is usually denoted  $\mathcal{A}_3$ , and our  $\mathcal{D}_2$  is not usually called a Coxeter group.

**PROOF.** This theorem follows from the preceding discussion, the transformation group of  $F$  comprising all automorphisms that fix the singular point  $x = 0$ , and perform no F-homotopy there. If  $\alpha \leftrightarrow \beta$ , i.e.,  $\theta_\infty \mapsto -\theta_\infty$ , is not explicitly included, then one may modify each automorphism with an odd number of

F-homotopies (negations) by following it by  $\theta_\infty \mapsto -\theta_\infty$ , to coerce even-signedness. This alteration is innocuous, since it leaves the FDE invariant.  $\square$

EXAMPLE ( $n = 3$ ). The correspondence between the elements of  $\mathcal{D}_2$ , the group of even-signed permutations of the nonzero singular points  $x = 1, \infty$  of the GHE, and transformations of  ${}_2F_1$  is

$$(13a) \quad [1_+][\infty_+] \sim {}_2F_1(\alpha, \beta; \gamma; x),$$

$$(13b) \quad [1_-][\infty_-] \sim (1-x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; x),$$

$$(13c) \quad [1_+\infty_+] \sim (1-x)^{-\alpha} {}_2F_1(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}),$$

$$(13d) \quad [1_-\infty_-] \sim (1-x)^{-\beta} {}_2F_1(\gamma-\alpha, \beta; \gamma; \frac{x}{x-1}).$$

Here (13b), (13c) are Euler's and Pfaff's transformations, and (13d) is a twisted Pfaff's transformation in which the first and second arguments, and also  $\alpha$  and  $\beta$ , are interchanged. Substituting any of (13b), (13c), (13d) for  $F(a, b; c; x)$  is an involutory operation. Any two of these three operations commute, and their product is the third, confirming that  $\mathcal{D}_2$  is isomorphic to the Klein four-group. If the transformation group of  ${}_2F_1$  is extended by explicitly including the involution  $\alpha \leftrightarrow \beta$ , the group becomes not  $\mathcal{D}_2$ , but  $\mathcal{B}_2$ . The nontrivial coset of  $\mathcal{D}_2$  in  $\mathcal{B}_2$  comprises

$$(14a) \quad [1_+][\infty_-] \sim {}_2F_1(\beta, \alpha; \gamma; x),$$

$$(14b) \quad [1_-][\infty_+] \sim (1-x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\beta, \gamma-\alpha; \gamma; x),$$

$$(14c) \quad [1_+\infty_-] \sim (1-x)^{-\beta} {}_2F_1(\beta, \gamma-\alpha; \gamma; \frac{x}{x-1}),$$

$$(14d) \quad [1_-\infty_+] \sim (1-x)^{-\alpha} {}_2F_1(\gamma-\beta, \alpha; \gamma; \frac{x}{x-1}).$$

Examination confirms that as expected, the set of transformations of  ${}_2F_1(x)$  in (13), (14), under composition, is isomorphic to the order-8 dihedral group.

The automorphism group of the GHE is isomorphic to  $\mathcal{D}_3 \cong S_4$  (of order 24), if  $\alpha \leftrightarrow \beta$  is not included. The group  $\mathcal{D}_2$  yielding (13a)–(13d) is a proper subgroup; a GHE automorphism may be more general than a transformation of  ${}_2F_1$ . In all,  $\mathcal{D}_3$  yields 24 solutions of the GHE, which are the well-known solutions of Kummer. They are partitioned into 6 equivalence classes of size 4, one of which is (13a)–(13d).

EXAMPLE ( $n = 4$ ). For any of the six homographies  $x' = M(x)$  given above, which in effect permute the nonzero singular points  $x = 1, a, \infty$  of the HE, one can rewrite (11) in HE notation, obtaining a Möbius transformation of  $HL$ . The painful part is computing the transformed accessory parameter. The usual parametrization (1b) of the HE is not adapted to this, and a closely related one is better.

First, write  $q = \alpha\beta Q$ , where  $Q$  is an alternative accessory parameter. ( $Q$ , not  $q$ , was Heun's accessory parameter; the present convention was introduced by Erdélyi.) Second, perform a change of variable,  $\bar{x} = a/x$ , i.e.,  $(0\infty)(1a)$ , accompanied by an F-homotopy that restores the transformed HE to HE form. In P-symbol terms,

$$(15) \quad x^\alpha \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{array} \right\} = \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & a/x \\ 0 & 0 & 0 & \alpha & \\ \beta-\alpha & 1-\epsilon & 1-\delta & \alpha-\gamma+1 & \end{array} \right\}.$$



That is, if  $F = v(x)$  is a solution of the HE (1b), then  $x^\alpha v(x)$ , regarded as a function of  $\bar{x} = a/x$ , i.e., as a multiple of  $\bar{x}^{-\alpha} v(a/\bar{x})$ , will satisfy a HE of the form

$$(16) \quad \left\{ D_{\bar{x}}^2 + \left[ \frac{\bar{\gamma}}{\bar{x}} + \frac{\bar{\delta}}{\bar{x}-1} + \frac{\bar{\epsilon}}{\bar{x}-a} \right] D_{\bar{x}} + \left[ \frac{\bar{\alpha}\bar{\beta}(\bar{x}-\bar{Q})}{\bar{x}(\bar{x}-1)(\bar{x}-a)} \right] \right\} F = 0,$$

with parameters  $\bar{\alpha} = \alpha$ ,  $\bar{\beta} = \alpha - \gamma + 1$ ,  $\bar{\gamma} = \alpha - \beta + 1$ ,  $\bar{\delta} = \epsilon$ ,  $\bar{\epsilon} = \delta$ . By direct computation,  $\bar{Q} = [\beta Q + (\epsilon - \beta)a + (\delta - \beta)] / (\alpha - \gamma + 1)$ .

In computing the Möbius transformations of  $Hl$ ,  $\bar{x} = a/x$  is a more natural independent variable than  $x$ , and  $\bar{Q}$  a more natural parameter than  $Q$  or  $q$ . This is because in terms of  $\bar{x}$ , each of the six homographies is *affine linear*, since it fixes the point  $\bar{x} = \infty$ , i.e.,  $x = 0$ . From each homography  $x' = M(x)$ , the affine map  $\bar{x}' = \bar{M}(\bar{x})$  follows from  $\bar{x}' = \bar{M}(\bar{x}) = a'/x' = a'/M(a/\bar{x})$ . The correspondence is

$$\begin{array}{llllll} M(x) : & x, & \frac{x}{x-1}, & \frac{(1-a)x}{x-a}, & \frac{(a-1)x}{a(x-1)}, & \frac{x}{a}, & \frac{x}{1-a} \\ \bar{M}(\bar{x}) : & \bar{x}, & \frac{a-\bar{x}}{a-1}, & 1-\bar{x}, & \frac{a-\bar{x}}{a}, & \frac{\bar{x}}{a}, & \frac{1-\bar{x}}{1-a}. \end{array}$$

And from (16), under any of these affine changes of variable  $\bar{x}' = \bar{M}(\bar{x})$ , it is clearly the case that  $\bar{Q}' = \bar{M}(\bar{Q})$ . So,  $\bar{Q}'$  in terms of  $\bar{Q}$  is quite easy to calculate.

The homography  $M(x) = \frac{x}{x-1}$ , which comes from the permutation  $(1\infty)(a)$  of the nonzero singular points  $x = 1, a, \infty$ , is typical. For this homography, Eq. (11) says that in a neighborhood of  $x = 0$ ,  $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$  can equivalently be written as  $(1-x)^{-\alpha} Hl(a', q'; \alpha', \beta'; \gamma', \delta'; \frac{x}{x-1})$ , in which  $a' = \frac{a}{a-1}$ ,  $\alpha' = \alpha$ ,  $\beta' = \alpha - \delta + 1$ ,  $\gamma' = \gamma$ , and  $\delta' = \alpha - \beta + 1$ , as is easily verified. What remains to be determined is the transformed parameter  $q'$ . But  $\bar{Q}' = (a - \bar{Q})/(a-1)$ , since the affine linear map associated to  $M$  is  $\bar{M}(\bar{x}) = \frac{a-\bar{x}}{a-1}$ . Taking into account that

$$\begin{aligned} \bar{Q} &= [\beta Q + (\epsilon - \beta)a + (\delta - \beta)] / (\alpha - \gamma + 1), \\ \bar{Q}' &= [\beta' Q' + (\epsilon' - \beta')a' + (\delta' - \beta')] / (\alpha' - \gamma' + 1), \end{aligned}$$

and that  $q = \alpha\beta Q$  and  $q' = \alpha'\beta'Q'$ , one finds  $q' = (-q + \gamma\alpha a)/(a-1)$ .

The preceding procedure can be carried out for all six homographies. It yields the following bijection between (positively signed) permutations of the nonzero singular points  $x = 1, a, \infty$  of the HE, and Möbius transformations of  $Hl$ .

$$\begin{aligned} [1_+][a_+][\infty_+] &\sim Hl(a, q; \alpha, \beta; \gamma, \delta; x), \\ [1_+\infty_+][a_+] &\sim (1-x)^{-\alpha} Hl\left(\frac{a}{a-1}, \frac{-q+\gamma\alpha a}{a-1}; \alpha, \alpha - \delta + 1; \gamma, \alpha - \beta + 1; \frac{x}{x-1}\right), \\ [1_+][a_+\infty_+] &\sim (1-\frac{x}{a})^{-\alpha} Hl(1-a, -q + \gamma\alpha; \alpha, -\beta + \gamma + \delta; \gamma, \delta; \frac{(1-a)x}{x-a}), \\ [1_+a_+][\infty_+] &\sim Hl(\frac{1}{a}, \frac{q}{a}; \alpha, \beta; \gamma, \alpha + \beta - \gamma - \delta + 1; \frac{x}{a}), \\ [1_+a_+\infty_+] &\sim (1-\frac{x}{a})^{-\alpha} Hl\left(\frac{1}{1-a}, \frac{q-\gamma\alpha}{a-1}; \alpha, -\beta + \gamma + \delta; \gamma, \alpha - \beta + 1; \frac{x}{x-a}\right), \\ [1_+\infty_+a_+] &\sim (1-x)^{-\alpha} Hl\left(\frac{a-1}{a}, \frac{-q+\gamma\alpha a}{a}; \alpha, \alpha - \delta + 1; \gamma, \alpha + \beta - \gamma - \delta + 1; \frac{(a-1)x}{a(x-1)}\right). \end{aligned}$$

Here, the second transformation is the just-derived one. It and the third are Pfaff-type transformations of  $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$ . The fourth is a simple ' $\delta \leftrightarrow \epsilon$ ' one, but the fifth and sixth are quite novel, having no  ${}_2F_1$  analogue. They cyclically permute the nonzero singular points, so they are not involutions: they are of order 3.

The full transformation group of  $Hl$  is isomorphic to  $\mathcal{D}_3 \cong S_4$ , or to  $\mathcal{B}_3 \cong \mathbb{Z}_2 \times S_4$  if the  $\alpha \leftrightarrow \beta$  interchange is included. It is generated by the Möbius

transformations, and by F-homotopic ones of the type (8), which are easier to work out explicitly (details are left to the reader). The simplest such is

$$[1_-][a_+][\infty_-] \sim (1-x)^{1-\delta} Hl(a, q-(\delta-1)\gamma a; \beta-\delta+1, \alpha-\delta+1; \gamma, 2-\delta; x),$$

coming from an F-homotopy at  $x = 1$ , which is analogous to (13b), Euler's transformation of  ${}_2F_1$ . An explicit bijection between the order-24 group  $\mathcal{D}_3$  and the transformations of  $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$  is given in [14, Table 2]. The 24 transformations were obtained by machine computation. But as the present treatment makes clear, they can also be obtained reliably by hand.

The automorphism group of the HE is isomorphic to  $\mathcal{D}_4$  (of order 192), or to  $\mathcal{B}_4$  (of order 384), if  $\alpha \leftrightarrow \beta$  is included. The entire set of 192 HE solutions appears in Ref. [14], and the bijection to  $\mathcal{D}_4$  is given explicitly. The 192 are partitioned into 8 equivalence classes, one of which comprises the just-mentioned 24 transformed versions of  $Hl$ . The remaining 7 come from the cosets of  $\mathcal{D}_3$  in  $\mathcal{D}_4$ . For later use, one of the  $168 = 192 - 24$  solutions of the latter sort will be given here; namely,

$$(17) \quad [a_+0_+1_+][\infty_+] \sim Hl\left(\frac{a-1}{a}, \frac{-q+\beta\alpha a}{a}; \alpha, \beta; \alpha+\beta-\gamma-\delta+1, \gamma; \frac{a-x}{a}\right).$$

This is obtained from a Möbius lifting  $M: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$ , where  $M(x) := \frac{a-x}{a}$ . Since  $M^{-1}(0) = a \neq 0$ , it is not a transformed version of  $Hl$ : it is a zero-exponent solution of the HE near the singular point  $x = a$ . It and  $Hl(a, q; \alpha, \beta; \gamma, \delta; x)$  do not equal each other on their common domain of definition, if this domain is nonempty.

#### 4. Quadratic and quartic transformations of $Hl$

The transformations of  $Hl$  derived in § 3 were based on Möbius transformations of  $\mathbb{P}^1$ , i.e., on degree-1 covering maps  $M: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$ . But if  $R: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$  is any rational map, a HE  $\mathcal{L}u = 0$  on the Riemann sphere  $\mathbb{P}_x^1$ , will lift along  $R$  to an FDE  $(R^*\mathcal{L})u = 0$  on  $\mathbb{P}_{x'}^1$ . The characteristic exponents of the operator  $R^*\mathcal{L}$  are determined by Prop. 2.1. If  $R^*\mathcal{L}$  has only four singular points, the lifted FDE will be a HE in its own right. Such a lifting is said to be a degree-(deg  $R$ ) transformation of the HE. If, moreover,  $R(0) = 0$ , then a formula relating a local solution (at  $x = 0$ ) of the form  $Hl(x)$ , to one of the form  $Hl(x') = Hl(R(x))$ , will result.

Erdélyi [8, §15.3] pointed out the existence of quadratic and quartic transformations of the HE, though he did not derive the corresponding functional equations satisfied by the special function  $Hl$ . He showed that if any two of the singular points  $x' = 0, 1, a', \infty$  of the HE on  $\mathbb{P}_{x'}^1$ , have exponent difference  $\frac{1}{2}$ , then there is a quadratic lifting to a HE on  $\mathbb{P}_x^1$ . (The singular points  $x = 0, 1, a, \infty$  of the lifted HE may differ, since  $a' \neq a$  in general.) For instance, one has

$$(18) \quad (1-x)^\alpha \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & 2\alpha & \\ 1-\gamma & \gamma-2\alpha & 1-\gamma & \gamma & \end{array} \right\} = \left\{ \begin{array}{cccc|c} 0 & 1 & a' & \infty & R(x) \\ 0 & 0 & 0 & \alpha & \\ 1-\gamma & \frac{1}{2} & \frac{1}{2} & \gamma-\alpha & \end{array} \right\},$$

in which the map  $x' = R(x)$  is of the form

$$(19) \quad R(x) = A \frac{x(a-x)}{1-x},$$

provided that  $a, a' \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$  are related by

$$(20) \quad a^2(1 - a')^2 - 16(1 - a)a' = 0,$$

and the ‘multiplier’  $A = A(a, a')$  is given by the formula

$$(21) \quad A = A(a, a') = \frac{1 + a'}{2(2 - a)}.$$

The P-symbol identity (18) follows from Prop. 2.1, since  $R^{-1}(0) = \{0, a\}$  and  $R^{-1}(\infty) = \{1, \infty\}$ ; and as one can check,  $x' = 1, a'$  are critical values of the map  $x' = R(x)$ , the corresponding critical points being mapped doubly to them. The latter requirement, which causes the singular points  $x' = 1, a'$  to ‘disappear’ when lifted to  $\mathbb{P}_x^1$ , is the source of the quadratic constraint (20) and the formula (21).

The Riemann surface defined by (20) is of genus zero. Equivalently,  $a, a'$  can be expressed in terms of a complex parameter  $t$ . By examination, one can choose

$$(22) \quad a(t) = \frac{t(t+8)}{(t+4)^2}, \quad a'(t) = \frac{t^2}{(t+8)^2}; \quad A(t) = \left(\frac{t+4}{t+8}\right)^2.$$

The choices  $t = 0, -4, -8, \infty$  are ‘unphysical’, since each corresponds to one of  $a, a'$  equaling one of  $0, 1, \infty$ . So, the possible quadratic Heun liftings of the form (18) are parametrized by the quadruply punctured sphere  $\mathbb{P}_t^1 \setminus \{0, -4, -8, \infty\}$ .

The quadratic (and biquadratic quartic) transformations of  $Hl$  can now be determined. To avoid degenerate cases, it will be assumed that the underlying rational map  $R: \mathbb{P}_x^1 \rightarrow \mathbb{P}_{x'}^1$  has the property that each critical value, on  $\mathbb{P}_{x'}^1$ , is one of the HE singular points  $x' = 0, 1, a, \infty$ .

**THEOREM 4.1.** *There is a unique quadratic transformation of  $Hl$ , up to pre- and post-composition with Möbius and F-homotopic transformations of  $Hl$ . It is*

$$\begin{aligned} &Hl(a, q; 2\alpha, \gamma; \gamma, 2\alpha - \gamma + 1; x) \\ &= (1 - x)^{-\alpha} Hl(a', A(q - \gamma\alpha a); \alpha, \gamma - \alpha; \gamma, \tfrac{1}{2}; R(x)), \end{aligned}$$

where  $a, a'$  are related by (20), and the quadratic map  $R$  and multiplier  $A$  are defined by (19), (21). This equality holds on a neighborhood of  $x = 0$ .

**PROOF.** Any quadratic map  $R$  must have branching schema  $1+1 = 1+1 = 2 = 2$ . (A schema of this sort, with four ‘slots’, lists the multiplicities with which the inverse images of the four singular points  $x = 0, 1, a', \infty$  on  $\mathbb{P}_{x'}^1$  are mapped to them; orderings of slots and inverse images are not significant.) In general, the lifted HE will have six singular points. The number will be reduced to four, by Prop. 2.1, if each of the two singular points on  $\mathbb{P}_{x'}^1$ , listed last has exponent difference  $\frac{1}{2}$ .

Let the schema be annotated by underscoring each inverse image that is a singular point of the operator  $R^*\mathcal{L}$  on  $\mathbb{P}_x^1$ . Then there is a unique annotated schema corresponding to a lifting of an HE to a HE, namely  $\underline{1}+\underline{1} = \underline{1}+\underline{1} = 2 = 2$ . For this to yield a transformation of  $Hl$ , one must have  $R(0) = 0$ . So  $x' = 0$  must be assigned to one of the first two slots. The remaining singular points  $x' = 1, a', \infty$  can be assigned arbitrarily to the remaining three slots; and the singular points  $x = 1, a, \infty$  to the remaining inverse images of type  $\underline{1}$ . Any such arrangement can be reduced by pre- and post-composition with Möbius transformations to the scheme (18), except that F-homotopic transformations may also be applied.

The  $Hl$  identity of the theorem can be read off from (18), except for the value  $q' = A(q - \gamma\alpha a)$  of the right-hand accessory parameter, which follows by direct computation: applying the change of variable  $x' = R(x)$  to the HE  $\mathcal{L}u = 0$ .  $\square$

REMARK 4.1.1. The full set of quadratic transformations of  $Hl$  is obtained by (i) composing with Möbius maps, i.e., using  $M_2 \circ R \circ M_1$  rather than  $R$  as the covering map (the left side of the identity being transformed by  $M_1^{-1}$ , and the right by  $M_2$ ), and (ii) acting on either side by F-homotopies. Quadratic transformations are therefore bijective to the elements of  $\mathcal{D}_3 \times \mathcal{D}_3$ , modulo an appropriate stabilizing subgroup. (Here ' $\alpha \leftrightarrow \beta$ ' is disregarded.) A complete list will appear elsewhere.

REMARK 4.1.2. To place Theorem 4.1 in context, there is no analogous uniqueness result for the quadratic transformations of the Gauss hypergeometric function  ${}_2F_1$ . There is a unique annotated schema for a quadratic lifting of a GHE on  $\mathbb{P}_{x'}^1$  to a GHE on  $\mathbb{P}_x^1$ ; namely,  $\underline{1} + \underline{1} = \underline{2} = 2$ . But there are two ways of assigning  $x' = 0$ : it can be assigned to the first slot or the second, and the resulting quadratic maps  $R(x)$  are distinct (up to pre- and post-composition with Möbius transformations, i.e., with Pfaff's involution  $x \mapsto x/(x-1)$ ). The two choices yield

$$\begin{aligned} R(x) &= 4x(1-x), \quad \frac{-4x}{(1-x)^2}, \quad \frac{4x}{(1+x)^2}, \quad \frac{4x(1-x)}{(1-2x)^2}; \\ R(x) &= \frac{x^2}{(2-x)^2}, \quad \frac{-x^2}{4(1-x)}, \end{aligned}$$

respectively. Taking pre- and post-composition with F-homotopies (i.e., with Euler's transformation) into account, one sees that the transformations coming from the first alternative are bijective to the elements of the group  $\mathcal{D}_2 \times \mathcal{D}_2$ , and those from the second are bijective to the same group, modulo a stabilizing subgroup of order 2. A partial table of quadratic transformations of  ${}_2F_1$  is given in [8, §2.11]. The existence of two distinct types has been stressed by Askey [3].

Erdélyi also pointed out that if *three* of the exponent-differences of the HE on  $\mathbb{P}_{x'}^1$  are equal to  $\frac{1}{2}$ , such as those at  $x' = 1, a', \infty$ , then it can be lifted to a HE along three distinct degree-2 covering maps  $x' = R(x)$ , and each can be followed by a second degree-2 lifting, leading to a *biquadratic quartic transformation*. By examination, any such composition of two quadratic maps,  $x' = S(x)$ , has branching schema  $\underline{1} + \underline{1} + \underline{1} + \underline{1} = 2 + 2 = 2 + 2 = 2 + 2$ . This schema is exemplified by

$$(23) \quad S(x) = \frac{4ax(1-x)(a-x)}{(a-x^2)^2} = 1 - \frac{(a-2ax+x^2)^2}{(a-x^2)^2} = a - \frac{a(a-2x+x^2)^2}{(a-x^2)^2},$$

for which  $a = a'$ ; for this map,  $S^{-1}(0) = \{0, 1, a, \infty\}$ . The schema is so symmetrical that when deriving quartic transformations of  $Hl$  from it, there is essentially only one way of assigning  $x' = 0, 1, a', \infty$  to the slots, and  $x = 0, 1, a, \infty$  to the inverse images of type  $\underline{1}$ . Proceeding as above, and computing the accessory parameter  $q$  of the lifted HE from  $q'$  by direct computation, one obtains the following.

THEOREM 4.2. *There is a unique biquadratic quartic transformation of  $Hl$ , up to pre- and post-composition with Möbius and F-homotopic transformations of  $Hl$ . Valid on a neighborhood of  $x = 0$ , it is*

$$\begin{aligned} &Hl(a, q; 2\gamma - 1, \gamma; \gamma, \gamma; x) \\ &= (1 - x^2/a)^{-\gamma + \frac{1}{2}} Hl\left(a, q/4; \frac{\gamma}{2} - \frac{1}{4}, \frac{\gamma}{2} + \frac{1}{4}; \gamma, \frac{1}{2}; S(x)\right), \end{aligned}$$

where the biquadratic quartic map  $S$  is defined by (23).

If  $\gamma = 1/2$ , then all four of the exponent-differences of the HE are equal to  $\frac{1}{2}$ . In this quite degenerate case, the biquadratic  $Hl$  identity specializes as follows.

**COROLLARY 4.2.1.** *The special function  $H(a, q; x) := Hl(a, q; 0, \frac{1}{2}, \frac{1}{2}; x)$ , where  $a \in \mathbb{C} \setminus \{0, 1\}$  and  $q \in \mathbb{C}$ , which is defined on  $|x| < \min(1, |a|)$ , satisfies the functional equation  $H(a, q; x) = H(a, q/4; S(x))$  on a neighborhood of  $x = 0$ .*

**REMARK.** By clever changes of variable, due to L. Carlitz (see Valent [21]), one can show  $H(a, q; x) = \cosh \left[ 2\sqrt{q/a} \operatorname{sn}^{-1}(\sqrt{x}) \right]$ , where  $\operatorname{sn}^{-1}$  is the inverse of the Jacobi elliptic function  $\operatorname{sn}(\cdot | 1/a)$ , and the branch that vanishes at  $x = 0$  is meant. So the corollary says that  $\operatorname{sn}^{-1}(\sqrt{x}) = \operatorname{sn}^{-1}(\sqrt{S(x)})/2$ , i.e., that  $\operatorname{sn}^2(2u | 1/a) = S(\operatorname{sn}^2(u | 1/a))$ . That is, the  $\gamma = 1/2$  case of the biquadratic  $Hl$  transformation is really the duplication formula for the Jacobi function [8, §13.17], in disguise. That the biquadratic quartic function  $S$  plays a dual role is not well known.

## 5. From $Hl$ to ${}_3F_2$

The local Heun function  $Hl(x) := \sum_{n=0}^{\infty} c(n)x^n$  has been less extensively investigated than the Gauss hypergeometric function  ${}_2F_1(x)$ , or even the generalized hypergeometric functions  ${}_3F_2(x)$ ,  ${}_4F_3(x)$ , etc. The presence of a fourth singular point  $x = a$  in the HE (1b), or equivalently, the fact that the recurrence (2b) satisfied by the coefficient sequence  $c: \mathbb{N} \rightarrow \mathbb{C}$  is of second order rather than first, is a novel and somewhat disconcerting feature. It is natural to ask whether, for certain parameter choices,  $Hl$  can be reduced to more familiar transcendental functions.

That  $Hl$  may reduce to  ${}_3F_2$  was shown by Letessier et al. [11]. In this section their result is rederived from scratch, starting from the ‘ $Hl$  side’ rather than the ‘ ${}_3F_2$  side,’ as it were. A connection to P-symbols will be made. As usual, it will be assumed that the parameter  $\gamma$  is not a nonpositive integer.

First, consider how the singular point  $x = a$  of (1b) can be removed or otherwise tamed. The simplest case is  $\epsilon = 0$ , when the characteristic exponents of  $x = a$  are  $0, 1 - \epsilon = 1$ , which are those of an ordinary point. (The case  $\epsilon = 2$ , which is related to this one by an F-homotopy, is left to the reader.) Since the difference between the two exponents is an integer, the local solution corresponding to the smaller one, zero, will generically be logarithmic. But a glance at (1b) reveals that if  $q = \alpha\beta a$  when  $\epsilon = 0$ , then the HE will reduce to the GHE. Since  $\delta = \alpha + \beta - \gamma - \epsilon + 1$ , one has the reduction  $Hl(a, \alpha\beta a; \alpha, \beta; \gamma, \alpha + \beta - \gamma + 1; x) = {}_2F_1(\alpha, \beta; \gamma; x)$ .

The next simplest case is  $\epsilon = -1$ , when the exponents of  $x = a$  are  $0, 2$ . (The case  $\epsilon = 3$ , related to this one by an F-homotopy, is also left to the reader.) As before, the zero-exponent local solution of the HE at  $x = a$  will generically be logarithmic. If it is not, then by definition  $x = a$  will be a (simple) *apparent* singular point: all local solutions of the HE at  $x = a$  will be analytic.

**THEOREM 5.1.** *At fixed  $\alpha, \beta, \gamma \in \mathbb{C}$ , the family of HEs with  $\epsilon = -1$  (and hence  $\delta = \alpha + \beta - \gamma + 2$ ), the singular point at  $x = a$  being required to be apparent, can be parametrized by an auxiliary parameter  $e \in \mathbb{C}$  according to*

$$a = \frac{e(e - \gamma + 1)}{(e - \alpha)(e - \beta)}, \quad q = \alpha\beta \frac{(e + 1)(e - \gamma + 1)}{(e - \alpha)(e - \beta)}.$$

**PROOF.** The singular point  $x = a$  will be apparent iff  $q$  satisfies a certain algebraic condition. It can be worked out by substituting  $F(x) = \sum_{n=0}^{\infty} \tilde{c}(n)(x - a)^n$  into (1b), with  $\tilde{c}(0) = 1$ , and solving for  $\tilde{c}(1), \tilde{c}(2)$ , etc. In calculating  $\tilde{c}(2)$  a division

by zero will occur, unless  $q$  satisfies the condition, in which case the equation for  $\tilde{c}(2)$  will be  $0 \cdot \tilde{c}(2) = 0$ . The condition is that the right side of this equation equal zero.

More efficiently, one may exploit the explicit formula (17) for the zero-exponent local solution of the HE at  $x = a$ . When  $\epsilon = -1$ , it reduces to

$$(24) \quad Hl(a', q'; \alpha', \beta'; \gamma', \delta'; x') := Hl\left(\frac{a-1}{a}, \frac{-q+\beta\alpha a}{a}; \alpha, \beta; -1, \gamma; \frac{a-x}{a}\right).$$

Since  $\gamma' = -1$ , this  $Hl$  has exponents  $0, 2$  at  $x' = \frac{a-x}{a} = 0$ , and is generically logarithmic there. The condition on  $q'$  for  $Hl(a', q'; \alpha', \beta'; -1, \delta'; x')$  to be non-logarithmic follows readily from the recurrence (2b). One finds that the equation determining  $c(2)$  is of the form  $0 \cdot c(2) = 0$ , and therefore has a solution, iff

$$(25) \quad q'^2 + [(\alpha' + \beta' - \delta' + 1) + (\delta' - 1)a']q' + \alpha'\beta'a' = 0.$$

Substituting the expressions for  $a', q'; \alpha', \beta'; \delta'$  from (24) into (25) yields

$$(26) \quad q^2 + [(\gamma - 1) - (2\alpha\beta + \alpha + \beta)a]q + \alpha\beta a[(\alpha\beta + \alpha + \beta + 1)a - \gamma] = 0,$$

which is the desired condition on  $q$ .

If  $\alpha, \beta, \gamma$  are fixed, Eq. (26) defines a curve in the  $a$ - $q$  plane. (In the complex domain, it is a two-sheeted Riemann surface over  $\mathbb{P}_a^1$ .) Being quadratic, it is of genus zero and can be uniformized by a  $\mathbb{P}^1$ -valued parameter, say  $e$ . By examination, the parametrizations  $a = a(e)$ ,  $q = q(e)$  supplied in the theorem will work.  $\square$

REMARK 5.1.1. The theorem says that at fixed  $\alpha, \beta, \gamma$ , the family of HEs for which  $x = a$  has exponents  $0, 2$ , but is a relatively ‘tame’ apparent singular point, is parametrized by the Riemann sphere  $\mathbb{P}_e^1$ . Generically this sphere is sextuply punctured, since  $e = 0, \gamma - 1$  yield  $a = 0$ ,  $e = \alpha\beta/(\alpha + \beta - \gamma + 1), \infty$  yield  $a = 1$ , and  $e = \alpha, \beta$  yield  $a = \infty$ ; and necessarily  $a \neq 0, 1, \infty$ .

DEFINITION 5.2. The special function  $G(x) := G(\alpha, \beta; \gamma; e; x)$  is defined to equal  $Hl(a, q; \alpha, \beta; \gamma; \delta; x)$  on a neighborhood of  $x = 0$ . Here  $\delta = \alpha + \beta - \gamma + 2$  (so that  $\epsilon = -1$ ), and  $a, q$  are given in terms of  $e$  (and  $\alpha, \beta, \gamma$ ) in Theorem 5.1.

How the parametrized local functions  $G(x), {}_3F_2(x)$  are related will now be explained. The following facts about  ${}_3F_2(x)$  will be needed. It can be defined as a solution of a canonical third-order differential equation, or as a sum  $\sum_{n=0}^{\infty} d(n)x^n$ , where  $d: \mathbb{N} \rightarrow \mathbb{C}$  satisfies a first-order recurrence [8, Ch. 4]. The differential equation is  $[xD \prod_{i=1}^2 (xD + b_i - 1) - x \prod_{i=1}^3 (xD + a_i)]F = 0$ ; or equivalently,

$$(27) \quad \left\{ D^3 + \left[ \frac{a_1 + a_2 + a_3 + 3}{x - 1} - \frac{b_1 + b_2 + 1}{x(x - 1)} \right] D^2 + \left[ \frac{a_1 a_2 + a_2 a_3 + a_3 a_1 + a_1 + a_2 + a_3 + 1}{x(x - 1)} - \frac{b_1 b_2}{x^2(x - 1)} \right] D + \left[ \frac{a_1 a_2 a_3}{x^2(x - 1)} \right] \right\} F = 0.$$

Here  $a_1, a_2, a_3; b_1, b_2$  are complex-valued parameters, and the equation is invariant under separate permutations of  $a_1, a_2, a_3$  and  $b_1, b_2$ . The singular points are  $x = 0, 1, \infty$ , with respective characteristic exponents  $0, 1 - b_1, 1 - b_2; 0, 1, s; a_1, a_2, a_3$ , where  $s := b_1 + b_2 - a_1 - a_2 - a_3$  is the ‘parametric excess.’ If either of  $b_1, b_2$  is a nonpositive integer, the local solution of (27) corresponding to the zero exponent at  $x = 0$  will generically be logarithmic; but if neither is a nonpositive integer, which will be assumed henceforth, the solution will be analytic on  $|x| < 1$ . When normalized to unity at  $x = 0$ , it is denoted  ${}_3F_2(x) := {}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$ .

Equation (27) is not the most general third-order (linear, homogeneous) FDE with the above exponents, but it is the most general one associated to a first-order recurrence. The coefficient sequence  $d: \mathbb{N} \rightarrow \mathbb{C}$  of  ${}_3F_2(x)$  satisfies

$$(28) \quad \left\{ (n+b_1)(n+b_2)(n+1)E - (n+a_1)(n+a_2)(n+a_3) \right\} d = 0,$$

initialized by  $d(0) = 1$ . Here  $E$  is the shift operator. This hypergeometric recurrence is of course the traditional way of defining  ${}_3F_2(x)$ .

**THEOREM 5.3.** *The local function  $G(\alpha, \beta; \gamma; e; x)$  can be expressed in terms of  ${}_3F_2$ , as  ${}_3F_2(\alpha, \beta, e+1; \gamma, e; x)$ .*

**PROOF.** It is readily verified that if  $\epsilon = -1$  (so that  $\delta = \alpha + \beta - \gamma + 2$ ), and  $a, q$  are given the values supplied in Theorem 5.1, then the second-order difference operator in the recurrence equation (2b), the normalized solution of which is the coefficient sequence  $c: \mathbb{N} \rightarrow \mathbb{C}$  in the series expansion of  $G$ , has the factorization

$$(29) \quad \frac{1}{n+e+1} \left\{ \frac{e(e-\gamma+1)}{(e-\alpha)(e-\beta)} E - 1 \right\} \\ \times \left\{ (n+\gamma)(n+e)(n+1)E - (n+\alpha)(n+\beta)(n+e+1) \right\}.$$

The right factor is the operator appearing in (28), with  $a_1, a_2, a_3; b_1, b_2$  set to  $\alpha, \beta, e+1; \gamma, e$ . So the hypergeometric sequence  $d: \mathbb{N} \rightarrow \mathbb{C}$  with these parameters is a solution of (2b). Its initial conditions are those of  $c$ , i.e.,  $d(0) = 1, d(-1) = 0$ . So  $c = d$ , and  $G(x) = {}_3F_2(x)$ .  $\square$

**ALTERNATIVE PROOF.** It is readily verified that if  $a_1, a_2, a_3; b_1, b_2$  are set to  $\alpha, \beta, e+1; \gamma, e$ , then the third-order differential operator in Eq. (27), the unique (normalized) analytic local solution of which is  ${}_3F_2$ , has the factorization

$$(30) \quad \left\{ D + \left[ \frac{e+1}{x} + \frac{1}{x-1} + \frac{1}{x-a} \right] \right\} \\ \times \left\{ D^2 + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right] D + \left[ \frac{\alpha\beta x - q}{x(x-1)(x-a)} \right] \right\},$$

in which  $\epsilon := -1$ ,  $\delta := \alpha + \beta - \gamma + 2$ , and  $a, q$  are defined as in Theorem 5.1. The right factor is the operator appearing in the HE (1b). So,  ${}_3F_2(x) = G(x)$ .  $\square$

**REMARK.** The operator factorization (29), resp. (30), can be viewed as taking place in the noncommutative polynomial ring generated by  $n, E$ , resp. by  $x, D$ . (The noncommutativity is specified by  $[E, n] = E$ , resp.  $[D, x] = 1$ .) Such factorizations, when they exist, can be found by noncommutative Gröbner basis techniques [15]. There are also specialized algorithms for finding hypergeometric-sequence solutions to recurrences, i.e., for finding first-order right factors of difference operators [6, 16]. But in general, it is difficult to characterize when a difference or differential operator will factor. If  $\epsilon = -1$  and  $(a, q)$  lies on the algebraic curve (26), then the second-order difference operator associated with  $Hl$  will factor, as shown in (29). Conversely, the factoring of the third-order differential operator associated with  ${}_3F_2(a_1, a_2, e+1; b_1, e; x)$ , as in (30), yields a parametrization of the  $a$ - $q$  curve.

**COROLLARY 5.3.1.**  *$G(\alpha, \beta; \gamma; e; x)$  can be expressed in terms of the Gauss function  ${}_2F_1$ , as  ${}_2F_1(\alpha, \beta; \gamma; x) + (\alpha\beta/\gamma e)x {}_2F_1(\alpha+1, \beta+1; \gamma+1; x)$ , or equivalently, in terms of  ${}_2F_1$  and its derivative as  $e^{-1}[xD + e] {}_2F_1(\alpha, \beta; \gamma; x)$ . It can also be written as  $[(e-\gamma+1)/e] {}_2F_1(\alpha, \beta; \gamma; x) + [(\gamma-1)/e] {}_2F_1(\alpha, \beta; \gamma-1; x)$ .*

PROOF. For any generalized hypergeometric function, the case when an upper parameter exceeds a lower one by an integer is special: it can be reduced. For instance [13, §5.2],  ${}_3F_2(a_1, a_2, e+1; b_1, e; x)$  can be written in terms of the Gauss function, as  ${}_2F_1(a_1, a_2; b_1; x) + (a_1 a_2 / b_1 e) x {}_2F_1(a_1 + 1, a_2 + 1; b_1 + 1; x)$ , i.e., as  $e^{-1}[xD + e] {}_2F_1(a_1, a_2; b_1; x)$ . This can be verified by comparing series expansions. It is also the case that the general function  $F := {}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$  satisfies the three-term contiguity relation

$$(31) \quad F(a_3+) = [(a_3 - b_1 + 1)/a_3] F + [(b_1 - 1)/a_3] F(b_1-),$$

where  $a_3+$  and  $b_1-$  indicate incrementing and decrementing of parameters [17]. Setting  $(a_1, a_2, a_3; b_1, b_2) = (\alpha, \beta, e; \gamma, e)$  yields the second representation.  $\square$

By Theorem 5.3, the function  $G(x)$  has both an  $Hl$  and a  ${}_3F_2$  representation. Their relationship is revealed by their P-symbols, which are respectively

$$(32) \quad \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 2 & \beta & \end{array} \right\}, \quad \left\{ \begin{array}{cccc|c} 0 & 1 & a & \infty & x \\ 0 & 0 & 0 & \alpha & \\ 1-\gamma & 1-\delta & 2 & \beta & \\ 1-e & 1 & 1 & e+1 & \end{array} \right\},$$

with  $\delta := \alpha + \beta - \gamma + 2$ . In the third-order P-symbol on the right, the ordinary point  $x = a$  is listed explicitly, with exponents 0, 1, 2. (Any order- $k$  FDE has exponents  $\{0, \dots, k-1\}$  at any finite ordinary point.) The final exponent row comes from the left-multiplication by a first-order differential operator that was performed in (30), but the two P-symbols are otherwise the same. Appending the ‘1’ to the  $x = a$  column converts  $x = a$  to an ordinary point, leaving only three singular points.

This reduction of  $Hl$  to  ${}_3F_2$  is therefore an example of *desingularization*: removing an apparent singular point from a differential operator by left-multiplying by an auxiliary, appropriately chosen operator. Such desingularizations may be constructed algorithmically, as Abramov et al. [1] show. They also show how to desingularize difference operators, which in a certain sense have singular points too. Any order- $d$  operator  $L := \sum_{j=0}^d p_j(n)E^j$ , where the  $p_j$  are polynomials with  $p_d \neq 0$ , may be viewed as acting on functions defined on  $\mathbb{C}_n$ , the entire complex  $n$ -plane. The difference equation  $LF = 0$  will have  $d$  independent solutions that are meromorphic on  $\mathbb{C}_n$  and analytic on a common left half-plane. The (leading) singular points of  $L$  on  $\mathbb{C}_n$  are the roots of the polynomial  $p_d(n-d)$ , by definition; they are possible pole locations. If  $n = n_0$  is a root, but none of the  $d$  solutions has a pole there, it is said to be apparent. A desingularized version of  $L$  can be obtained by, e.g., left-multiplying  $L$  by an appropriate difference operator.

The right factor in (29), the first-order difference operator associated to the function  ${}_3F_2(\alpha, \beta, e+1; \gamma, e; x)$ , has singular points at  $n = 0, 1-\gamma, 1-e$ . It can be shown that the one at  $n = 1-e$  is merely apparent. The left-multiplication in (29) removes this apparent singular point. So, the reduction of this  ${}_3F_2$  to  $Hl$  is also a desingularization. The two desingularizations are dual to each other.

## 6. ${}_3F_2$ transformations

The generalized hypergeometric function  ${}_3F_2 := {}_3F_2(a_1, a_2, a_3; b_1, b_2; x)$  satisfies many identities, but its theory is less developed than that of  ${}_2F_1$ . It was shown



in §5 that the *ad hoc* special function

$$\begin{aligned} G(\alpha, \beta; \gamma; e; x) &= {}_3F_2(\alpha, \beta, e+1; \gamma, e; x) \\ &= e^{-1} [xD + e] {}_2F_1(\alpha, \beta; \gamma; x) \end{aligned}$$

has a Heun representation. As a result, certain transformation laws for  $G$  (and hence for a restricted  ${}_3F_2$ ) can be deduced from those for  $Hl$ .

Pfaff and Euler-like transformations of this  ${}_3F_2$  are given in Theorem 6.1 below, which is the main result of this final section. Each transformation has four free parameters and a nonlinear parametric constraint. Part (1) of the theorem subsumes several interesting identities discovered by Bailey and Slater, which have fewer free parameters. They appear as corollaries of the theorem.

Some terminology from general hypergeometric summation theory will be used. The function  ${}_r F_r(x) := {}_{r+1}F_r(a_1, \dots, a_{r+1}; b_1, \dots, b_r; x)$  can be defined as the solution of a canonical order- $(r+1)$  FDE, or as a sum  $\sum_{n=0}^{\infty} d(n)x^n$  that converges on  $|x| < 1$ . The FDE is  $[xD \prod_{i=1}^r (xD + b_i - 1) - x \prod_{i=1}^{r+1} (xD + a_i)]F = 0$ , and  $d: \mathbb{N} \rightarrow \mathbb{C}$  satisfies  $[(n+1) \prod_{i=1}^r (n + b_i)E - \prod_{i=1}^{r+1} (n + a_i)]d = 0$ .

In general, if  $\{a_i\}_{i=1}^{r+1}$  and  $\{b_i\}_{i=1}^r$  can be separately permuted so that  $a_1 + 1 = a_2 + b_1 = \dots = a_{r+1} + b_r$ , then the function  ${}_r F_r$  is said to be *well poised*. If each of these  $r+1$  equalities holds except the last, it is *nearly well poised*, or simply *nearly poised*. If  ${}_r F_r$  is well poised, resp. nearly poised, and also one of the  $\{b_i\}_{i=1}^r$ , resp.  $\{b_i\}_{i=1}^{r-1}$ , equals  $a_1/2$ , it is *very well poised*, resp. *nearly very well poised*.

THEOREM 6.1.

(1) *The function  ${}_3F_2$  satisfies the Pfaff-like identity*

$$\begin{aligned} &{}_3F_2(a_1, a_2, e+1; b_1, e; x) \\ &= (1-x)^{-a_1} {}_3F_2(a_1, b_1 - a_2 - 1, e' + 1; b_1, e'; \frac{x}{x-1}), \end{aligned}$$

*in which the  $e \leftrightarrow e'$  correspondence is*

$$e' = \frac{(b_1 - a_2 - 1)e}{e - a_2}.$$

(2) *The function  ${}_3F_2$  satisfies the Euler-like identity*

$$\begin{aligned} &{}_3F_2(a_1, a_2, e+1; b_1, e; x) \\ &= (1-x)^{b_1 - a_1 - a_2 - 1} {}_3F_2(b_1 - a_1 - 1, b_1 - a_2 - 1, e'' + 1; b_1, e''; x), \end{aligned}$$

*in which the  $e \leftrightarrow e''$  correspondence is*

$$e'' = \frac{(b_1 - a_1 - 1)(b_1 - a_2 - 1)e}{(b_1 - a_1 - a_2 - 1)e + a_1 a_2}.$$

Here  $e, e'$ , resp.  $e, e''$ , are taken to be finite. Also, the Möbius transformation  $e \mapsto e'$ , resp.  $e \mapsto e''$ , which is of the lower triangular form  $e \mapsto \frac{Ae}{Ce+D}$ , is assumed to be nonsingular, which will be the case if all upper parameters are nonzero, i.e., if neither  ${}_3F_2$  is identically equal to unity.

PROOF. The transformations of  $Hl(x) := Hl(a, q; \alpha, \beta; \gamma, \delta; x)$  analogous to Pfaff's and Euler's transformations of  ${}_2F_1$  are, from §3,

$$\begin{aligned} [1+\infty_+][a_+] &\sim (1-x)^{-\alpha} Hl(\frac{a}{a-1}, \frac{-q+\gamma\alpha a}{a-1}; \alpha, \alpha - \delta + 1; \gamma, \alpha - \beta + 1; \frac{x}{x-1}), \\ [1-][a_+][\infty_-] &\sim (1-x)^{1-\delta} Hl(a, q - (\delta-1)\gamma a; \beta - \delta + 1, \alpha - \delta + 1; \gamma, 2 - \delta; x). \end{aligned}$$

Theorem 5.1 gives the parametrization  $(a, q) = (a(e), q(e))$  of the case when the singular point  $x = a$  is a (simple) apparent one. In this case,  $Hl(x)$  equals  $G(\alpha, \beta; \gamma; e; x) = {}_3F_2(\alpha, \beta, e + 1; \gamma, e; x)$ . The transformed versions of  $Hl(x)$  have similar parametrizations  $(a', q') = (a'(e'), q'(e'))$  and  $(a'', q'') = (a''(e''), q''(e''))$ , and may likewise be written in terms of  ${}_3F_2$ . Here  $a' = \frac{a}{a-1}$ ,  $q' = \frac{-q+\gamma\alpha a}{a-1}$ , and  $a'' = a$ ,  $q'' = q - (\delta - 1)\gamma a$ . The transformations therefore yield isomorphisms between genus-zero algebraic curves,  $(a, q) \leftrightarrow (a', q')$  and  $(a, q) \leftrightarrow (a'', q'')$ , which on the parametric level are performed by Möbius correspondences  $e \leftrightarrow e'$  and  $e \leftrightarrow e''$ . Computing these explicitly yields the formulas in the theorem.  $\square$

REMARK. The transformations of the theorem can also be verified by expressing each  ${}_3F_2$  in terms of two  ${}_2F_1$ 's, as in Corollary 5.3.1, and exploiting results on  ${}_2F_1$ .

The Pfaff and Euler-like transformations generate an order-4 group of transformations of the family of  ${}_3F_2$ 's of the form  ${}_3F_2(a_1, a_2, e + 1; b_1, e; x)$ . The group is isomorphic to  $\mathcal{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . (The third nontrivial element of this group is a twisted Pfaff transformation, obtained from the Pfaff-like one by interchanging  $a_1, a_2$ .) If  $a_1 \leftrightarrow a_2$  is included explicitly, the resulting group will be isomorphic to  $\mathcal{B}_2$ , i.e., to the dihedral group of order 8. This group structure, familiar from  ${}_2F_1$ , is inherited from the automorphism group of the HE.

COROLLARY 6.1.1. *The function  ${}_3F_2$  satisfies the Pfaff-like identity*

$$\begin{aligned} & {}_3F_2(a_1, a_2, -a_2 + 1; b_1, -a_2; x) \\ &= (1-x)^{-a_1} {}_3F_2(a_1, b_1 - a_2 - 1, \frac{b_1 - a_2 + 1}{2}; b_1, \frac{b_1 - a_2 - 1}{2}; \frac{x}{x-1}). \end{aligned}$$

PROOF. Set  $e = -a_2$  in Theorem 6.1(1).  $\square$

REMARK. This special case, with three free parameters and no nonlinear constraint, was discovered by Bailey and Slater [19, (2.4.2.8)] and rediscovered by Baweja [4]. Their derivations were altogether different. The right-hand  ${}_3F_2$  is nearly very well poised, as one sees by permuting parameters.

COROLLARY 6.1.2. *There is a three-parameter reduction of  ${}_3F_2$  to a single  ${}_2F_1$ ; namely,*

$${}_3F_2(a_1, a_2, e + 1; b_1, e; x) = (1-x) {}_2F_1(a_1 + 1, a_2 + 1; b_1; x),$$

in which  $e := e(a_1, a_2; b_1) = a_1 a_2 / (a_1 + a_2 - b_1 + 1)$ .

PROOF. If  $e = a_1 a_2 / (a_1 + a_2 - b_1 + 1)$  then  $e' = a_1$ , reducing the right-hand side in Theorem 6.1(1) to  $(1-x)^{-a_1} {}_2F_1(a_1 + 1, b_1 - a_2 - 1; b_1; \frac{x}{x-1})$ . To obtain the given right-hand side, use Eq. (12), Pfaff's transformation of  ${}_2F_1$ .  $\square$

COROLLARY 6.1.2.1. *Any very well poised  ${}_3F_2$  can be reduced to a well poised  ${}_2F_1$ , according to*

$${}_3F_2(\alpha, \beta, \alpha/2 + 1; \alpha - \beta + 1, \alpha/2; x) = (1-x) {}_2F_1(\alpha + 1, \beta + 1; \alpha - \beta + 1; x).$$

PROOF. Set  $(a_1, a_2; b_1) = (\alpha, \beta; \alpha - \beta + 1)$  in Corollary 6.1.2.  $\square$

REMARK. This reduction of a very well poised  ${}_3F_2$  was also found by Bailey. It is sometimes reproduced in a different but equivalent form [19, (2.4.2.10)], in which Eq. (10), Euler's transformation of  ${}_2F_1$ , has been applied to the right side.

COROLLARY 6.1.2.2. *For any  $s, t \in \mathbb{C}$ , the parametrized function*

$$(1-x)^{2\alpha-1} {}_3F_2\left(2\alpha-1, (1-s)\alpha-(1+s)\beta-(\tfrac{1}{2}-t), e+1; (1-s)\alpha+(1-s)\beta-(\tfrac{1}{2}-t), e; x\right),$$

in which

$$e := (\alpha - \tfrac{1}{2}) \left[ \frac{(1-s)\alpha - (1+s)\beta - (\tfrac{1}{2}-t)}{\alpha - \beta - \tfrac{1}{2}} \right],$$

is stable under the interchange  $\alpha \leftrightarrow \beta$ .

PROOF. By Corollary 6.1.2, this function reduces to

$$(1-x)^{2\alpha} {}_2F_1\left(2\alpha, (1-s)\alpha - (1+s)\beta + (\tfrac{1}{2}+t); (1-s)\alpha + (1-s)\beta + (\tfrac{1}{2}+t); x\right).$$

Applying Eq. (10), Euler's transformation of  ${}_2F_1$ , has the effect of interchanging  $\alpha$  and  $\beta$  in this expression.  $\square$

REMARK. For each  $s, t \in \mathbb{C}$ , the invariance under  $\alpha \leftrightarrow \beta$  is really a special case of Theorem 6.1(2), the general Euler-like transformation of  ${}_3F_2(a_1, a_2, e+1; b_1, e; x)$ .

COROLLARY 6.1.2.3. *There is an involutory transformation of the two-parameter family of very well poised  ${}_3F_2$ 's, namely the  $\alpha \leftrightarrow \beta$  interchange*

$$\begin{aligned} (1-x)^{2\alpha-1} {}_3F_2(2\alpha-1, \alpha-\beta-\tfrac{1}{2}, \alpha+\tfrac{1}{2}; \alpha+\beta+\tfrac{1}{2}, \alpha-\tfrac{1}{2}; x) \\ = (1-x)^{2\beta-1} {}_3F_2(2\beta-1, \beta-\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2}; \beta+\alpha+\tfrac{1}{2}, \beta-\tfrac{1}{2}; x). \end{aligned}$$

PROOF. Set  $s = t = 0$  in Corollary 6.1.2.2.  $\square$

REMARK. This transformation law was also discovered by Bailey (see [8, §4.5]). It can be derived directly from Corollary 6.1.2.1, which reduces both sides to the same well poised  ${}_2F_1$ . The generalization given in Corollary 6.1.2.2 is new, as is its interpretation as a transformation of Euler type, a special case of Theorem 6.1(2).

It should be noted that the two sides of this transformation formula are solutions of different third-order differential equations. The solution spaces of these equations have nonempty intersection, but are distinct. This phenomenon is clarified by the respective P-symbols, which by examination are

$$\left\{ \begin{array}{ccc|c} 0 & 1 & \infty & x \\ 0 & 2\alpha & 0 & \\ \hline \tfrac{1}{2} - \alpha - \beta & 2\beta & \tfrac{1}{2} - \alpha - \beta & \\ \hline \tfrac{3}{2} - \alpha & 2\alpha - 1 & \tfrac{3}{2} - \alpha & \end{array} \right\}, \quad \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & x \\ 0 & 2\alpha & 0 & \\ \hline \tfrac{1}{2} - \alpha - \beta & 2\beta & \tfrac{1}{2} - \alpha - \beta & \\ \hline \tfrac{3}{2} - \beta & 2\beta - 1 & \tfrac{3}{2} - \beta & \end{array} \right\}.$$

The first two rows of exponents are the same in both, but the third row differs. This partial agreement is what one expects to see in two third-order operators that are distinct left multiples of a common second-order operator.

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